On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes

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Abstract

We prove the scale invariant Harnack inequality and regularity properties for harmonic functions with respect to an isotropic unimodal Lévy process with the characteristic exponent ψ satisfying some scaling condition. We show sharp estimates of the potential measure and capacity of balls, and further, under the assumption of that ψ satisfies the lower scaling condition, sharp estimates of the potential kernel of the underlying process. This allow us to establish the Krylov-Safonov type estimate, which is the key ingredient in the approach of Bass and Levin, that we follow.

1 Introduction

Let X_t be a Lévy process with the characteristic exponent

$$\psi(x) = \langle x, Ax \rangle - i \langle x, \gamma \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z \rangle} - 1 - i \langle x, z \rangle \mathbf{1}_{|z|<1} \right) \nu(dz), \quad z \in \mathbb{R}^d,$$

where A is a symmetric and non-negative definite matrix, ν is a Lévy measure, i.e. $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \, \nu(dz) < \infty$ and $\gamma \in \mathbb{R}^d$. A generator of this process has the following form, for $f \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{A}f(x) = \sum_{j,k} A_{jk}\partial_{jk}^2 f(x) + \sum_k \gamma_k \partial_k f(x) + \int \left(f(x+z) - f(x) - \mathbf{1}_{|z|<1} \langle z, \nabla f(x) \rangle \right) \nu(dz). \quad (1)$$

The *first exit time* of an (open) set $D \subset \mathbb{R}^d$ and *first hitting time* to D^c by the process X_t is defined by the formula

$$\tau_D = \inf\{t > 0; \, X_t \notin D\}, \qquad T_{D^c} = \inf\{t > 0 : X_t \in D^c\}.$$

A function $f : \mathbb{R}^d \to [0, \infty)$ is said that is harmonic with respect to X_t in an open set D if for any bounded open set B such that $\bar{B} \subset D$

$$f(x) = E^x f(X_{\tau_B}), \qquad x \in B.$$

The *scale invariant Harnack inequality* holds for a process X_t if for any $R > 0$ there exists a constant $C = C(R)$ such that for any function non-negative on $\mathbb{R}^d$ and harmonic in a ball $B(0, r)$, $r \leqslant R$,

$$\sup_{x \in B(0,r/2)} h(x) \leqslant C \inf_{x \in B(0,r/2)} h(x).$$

We say that the *global* scale invariant Harnack inequality holds if constant in the above inequality does not depend on R.

A measure $m(dx)$ is isotropic unimodal if there exists a non-increasing function $m_0 : (0, \infty) \to [0, \infty)$ such that $m(dx) = m_0(|x|)dx$, for $x \neq 0$. A process is isotropic unimodal if a transition probability $P_t(dx)$ is isotropic unimodal, for all $t > 0$.

Important class of isotropic unimodal Lévy processes are subordinate Brownian motions.

Let f be a positive function on $\mathbb{R}^d \setminus \{0\}$. We say that f satisfies the *weak lower scaling condition* WLSC(β, θ, C), if $\beta > 0$, $\theta \geqslant 0$, $C > 0$, and

$$f(\lambda x) \geq C \lambda^\beta f(x), \quad \text{for} \quad \lambda \geq 1, \quad |x| \geqslant \theta.$$

If f satisfies WLSC$(\beta, 0, C)$, then we say that f satisfies the *global* weak lower scaling condition.

The main purpose of this paper is to prove the scale invariant Harnack inequality and regularity properties for harmonic functions with respect to an isotropic unimodal Lévy process with the characteristic exponent satisfying the weak lower scaling condition. Our main technical results are sharp estimates of the potential measure and capacity of balls, and further, under the assumption of that ψ satisfies the weak lower scaling condition, sharp estimates of the potential kernel of the underlying process. This allow us to establish the Krylov-Safonov type estimate (see Proposition 25), which says that there are c and $\lambda < 1$ such that for a closed set $A \subset B(0, \lambda r)$,

$$P^x(T_A < \tau_{B(0,r)}) \geqslant c \frac{|A|}{|B(0,r)|}, \qquad x \in B(0, \lambda r).$$

This estimate is the key ingredient of the proofs of the Harnack inequality and local Hölder continuity of harmonic functions in the approach of Bass and Levin ([2]) that we follow.

Our main contribution is the fact that we assume only a mild condition for the characteristic exponent. Usually in the existing literature on the Harnack inequality for Lévy processes the assumptions were given in terms of the behaviour of the Lévy measure (see [22], Section 3). Our result seems to be important for application to subordinate Brownian motions. There are examples when the characteristic exponent is known, while estimates for the Lévy measure are not. We should also notice that our approach allows to deal with isotropic unimodal processes with the Lévy-Khinchine exponent behaving at infinity almost like the exponent for a Brownian motion, which to our best knowledge were not treated in the literature, except a few particular cases. Namely, we can take $\psi(x) = |x|^2 l(|x|)$, where l is slowly varying and goes to 0 at infinity. An example of such a process is for instance a process with density of its Lévy measure equal to $|x|^{-d-2} \log^{-2}(2 + |x|^{-1})$. Moreover, our result allows to extend the scale invariant Harnack inequality to its global version for many processes. For instance we get the global scale invariant Harnack inequality for α-stable relativistic processes.

The main results of this paper are following two theorems. The first one is the scale invariant Harnack inequality.

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Theorem 1. *Let $d \geqslant 3$. If ψ satisfies $WLSC(\beta, \theta, C)$, then the scale invariant Harnack inequality holds. Moreover, if ψ satisfies the global weak lower scaling condition, then the global scale invariant Harnack inequality holds.*

The next theorem deals with regularity of harmonic functions

Theorem 2. *Let $d \geqslant 3$ and ψ satisfy $WLSC(\beta, \theta, C)$. For any $R > 0$ there exist constants $c = c(R)$ and $\delta > 0$ such that, for any $0 < r \leqslant R$, and any bounded function h, which is harmonic in $B(0, r)$,*

$$|h(x) - h(y)| \leqslant c||h||_\infty \left(\frac{|x-y|}{r}\right)^\delta, \qquad x, y \in B(0, r/2).$$

Recently there has been a lot of research concerning non-local operators. For instance, the paper [6] established the scale-invariant finite range parabolic Harnack inequality for a class of jump-type Markov processes on metric measure spaces. A class of special subordinate Brownian motions have been studied in [14], where bounds for the densities of Lévy measure and potential measure and Harnack inequalities were established. Harnack inequalities and regularity estimates for harmonic function with respect to diffusion with jumps are proved in [9]. Related work on discontinuous processes include [22], [21], [16], [7] and [8]. Therefore it is pertinent to comment on the differences between our results and those of some related papers. For the sake of comparison we present them in the context of Lévy processes, however most of them are in a more general setting of Markov processes.

- One of the main assumptions in [6] is that the density of the Lévy measure is comparable to $\frac{1}{f(|x|)|x|^d}$ on $B(0, 1)$, where f is a strictly increasing continuous function and satisfies the following conditions. There exist $0 < \beta_1 \leqslant \beta_2$, and a constant c such that

$$c^{-1}\left(\frac{r_2}{r_1}\right)^{\beta_1} \leqslant \frac{f(r_2)}{f(r_1)} \leqslant c\left(\frac{r_2}{r_1}\right)^{\beta_2}, \qquad 0 < r_1 < r_2 < \infty,$$

$$\int_0^r \frac{s}{f(s)}ds \leqslant c\frac{r^2}{f(r)}, \qquad r > 0.$$

  One can easily check that the lower scaling condition for f implies the weak lower scaling condition for the characteristic exponent, hence our assumption is much weaker than that from [6]. In [7], under the assumption that the above estimate for the density of Lévy measure holds on the whole space, the authors obtained the global parabolic Harnack inequality. In our context of isotropic unimodal Lévy processes the global lower scaling for f is sufficient to get the global Harnack inequality (see Example 2 in subsection 3.4).

- In [22] the following Krylov-Safonov type estimate (Lemma 3.4) was derived

$$P^x(T_A < \tau_{B(0,r)}) \geqslant c\frac{\nu(4x)}{\int(1 \wedge |z|^2)\nu(z)dz}|A|.$$

  Such an estimate is sufficient in the proof of the Harnack inequality only if a density of Lévy measure satisfies similar conditions as in [6]. However, it will not work for $\nu(x) = \frac{1}{|x|^{d+2}\ln^2(2+|x|^{-1})}$ since applying it one obtains $P^x(T_A < \tau_B) \geqslant \frac{c}{\ln r^{-1}}\frac{|A|}{|B_r|}$, for $r \leqslant 1/2$. Hence if r goes to 0 the term $\frac{c}{\ln r^{-1}}$ vanishes, which makes the above bound useless for the proof of the scale invariant Harnack inequality.

- In [14] it was considered a class of special subordinate Brownian motions such that a subordinator has a non-increasing density of the Lévy measure. Moreover, there was some scaling assumption on the Laplace exponent of subordinator in terms of its derivative.

  In the present paper the weak lower scaling condition for the Laplace exponent of the subordinator is sufficient to obtain the Harnack inequality and we do not need to assume anything else about the Lévy measure of the subordinator. This does not mean that our result covers all the results of [14]. Their proof is not based on the Krylov-Safonov type estimate and it works for a large class of slowly varying Laplace exponents, while our approach does not cover that case. This is due to the fact that the Krylov-Safonov type estimate does not need to hold for the subordinate Brownian motions driven by subordinators with slowly varying Laplace exponents. On the other hand our results improve the results from [16], where it was studied only a particular case of subordinate Brownian motion with $\psi(x) = \frac{|x|^2}{\ln(1+|x|^2)} - 1$.

- Since we do not exclude a case when a Gaussian part is non-zero we mention the paper [8], where diffusions with jumps were considered. In this paper the density of the Lévy measure is assumed to be bounded from above $\nu(x) \leqslant c|x|^{-d-\alpha}$, for $|x| \leqslant 1$, where $\alpha \in (0, 2)$. Hence the result can not be applied to the process with $A = Id$ and $\nu(x) = \frac{1}{|x|^{d+2}\ln^2(2+|x|^{-1})}$. Notice that for any Lévy process with a non-trivial Gaussian part (rank $A = d$) the WLSC property holds for the characteristic exponent.

- In [9] it is assumed that, for any $r < 1$, there exist constants c and α such that $\nu(x - z) \leqslant cr^{-\alpha}\nu(y - z)$, for $|x - y| < r$ and $|x - z| > r$. Therefore for instance this result does not cover the case $A = Id$ and $\nu(x) = \frac{1}{|x|^{d+2}\ln^2(1+|x|^{-1})}e^{-|x|^2}$, for which we even have the global Harnack inequality, due to Theorem 1, since $\psi(x) \approx |x|^2$.

The paper is organized as follows. In Section 2, we give some preliminary results for general Lévy processes. Section 3 is devoted to prove estimates of Green function and the main results. Moreover, several examples are presented to which our approach applies. In Section 4 some conditions are stated that are sufficient to prove the scale invariant Harnack inequality for Lévy processes not necessarily isotropic and unimodal.

2 Preliminaries

In this section we introduce notation and prove some auxiliary results for general Lévy processes. We denote incomplete Gamma functions by

$$\gamma(\delta, t) = \int_0^t e^{-u}u^{\delta-1}du, \qquad \Gamma(\delta, t) = \int_t^\infty e^{-u}u^{\delta-1}du, \quad \delta, \, t > 0.$$

Let $B(x, r)$ denote a ball of center x and radius $r > 0$ and let $B_r = B(0, r)$. By $\mathcal{L}$ we denote the Laplace transform, that means, for a measure μ on $[0, \infty)$,

$$\mathcal{L}\mu(\lambda) = \int_0^\infty e^{-\lambda s}\mu(ds), \quad \lambda \geqslant 0.$$

For two non-negative functions f and g we write $f(x) \approx g(x)$ if there is a positive number C (i.e. a constant) such that $C^{-1}f(x) \leqslant g(x) \leqslant Cf(x)$. This C is called a comparability constant.

We write $C = C(a, \ldots, z)$ to emphasize that C depends only on $a, \ldots, z$. An integral $\int_a^b \ldots$ we understand as $\int_{[a,b)} \ldots$.

Our primary object is a potential measure G, which is well defined for a transient process, by the following formula

$$G(x, A) = \int_0^\infty P^x(X_t \in A)dt = E^x \int_0^\infty \mathbf{1}_A(X_t)dt,$$

where A is a Borel subset of $\mathbb{R}^d$. In what follows we always consider Borel subsets of $\mathbb{R}^d$ without further mention. Let $G(A) = G(0, A)$. Notice that $G(x, A) = G(A - x)$. By a slight abuse of notation we also use G to denote the density of the absolutely continuous (with respect to the Lebesque measure) part of the potential measure and then we call $G(x, y) = G(y - x)$ a potential kernel.

The fundamental object of the potential theory is the *killed process* X_t^D when exiting the set D. It is defined in terms of sample paths up to time τ_D. More precisely, we have the following "change of variables" formula:

$$E^x f(X_t^D) = E^x[t < \tau_D; f(X_t)], \quad t > 0.$$

The potential measure of the process X_t^D is called the *Green measure* and is denoted by G_D. That is

$$G_D(x, A) = E^x \int_0^{\tau_D} \mathbf{1}_A(X_t)dt.$$

The corresponding kernel will be called the *Green function* of the set D and denoted $G_D(x, y)$. If the potential measure is absolutely continuous, then we have

$$G_D(x, y) = G(y - x) - E^x G(X_{\tau_D} - y). \tag{2}$$

Another important object in the potential theory of X_t is the *harmonic measure* of the set D. It is defined by the formula:

$$P_D(x, A) = E^x[\tau_D < \infty; \mathbf{1}_A(X_{\tau_D})].$$

The density kernel (with respect to the Lebesgue measure) of the measure $P_D(x, A)$ (if it exists) is called the *Poisson kernel* of the set D. The relationship between the Green function of D and the harmonic measure is provided by the Ikeda-Watanabe formula [12],

$$P_D(x, A) = \int_D \nu(A - y)G_D(x, dy), \quad A \subset (\bar{D})^c. \tag{3}$$

Important examples of isotropic unimodal Lévy processes are subordinate Brownian motions and some our results are restricted to this class of processes. By T_t we denote a subordinator i.e. a non-decreasing Lévy process starting from 0. The Laplace transform of T_t is of the form

$$Ee^{-\lambda T_t} = e^{-t\phi(\lambda)}, \quad \lambda \geqslant 0,$$

where ϕ is called the Laplace exponent of T. ϕ is a Bernstein function and has the following representation:

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda u})\mu(du),$$

where $b \geq 0$ and μ is a Lévy measure on $(0, \infty)$ such that $\int (1 \wedge u) \mu(du) < \infty$.

The potential measure of the subordinator T is denoted by U. Its Laplace transform is equal to

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda s} U(ds) = \frac{1}{\phi(\lambda)}. \tag{4}$$

We say that a Bernstein function ϕ is special if there exists a decreasing positive density u on $(0, \infty)$ of a measure $U|_{(0,\infty)}$. For a different characterization of special Bernstein functions see e.g. [20].

Let B_t be a Brownian motion in $\mathbb{R}^d$ with the characteristic function of the form

$$Ee^{i\xi B_t} = e^{-t|x|^2}, \quad x \in \mathbb{R}^d.$$

By $g_t(x)$ we denote the transition density of B_t. Assume that B_t and T_t are stochastically independent. Then the process $X_t = B_{T_t}$ defines a subordinate Brownian motion. It is clear that the characteristic function of X_t takes the form

$$Ee^{i\xi X_t} = e^{-t\phi(|x|^2)}, \quad x \in \mathbb{R}^d.$$

The Lévy measure of the process X_t is given by the following formula for its density

$$\nu(x)dx = \int_0^\infty g_u(x)\mu(du)dx,$$

while its potential measure is equal to

$$G(A) = \frac{1}{\lim_{\lambda\to\infty}\phi(\lambda)}\mathbf{1}_{\{0\}}(A) + \int_A \int_0^\infty g_s(y)U(ds)dy. \tag{5}$$

A subordinator which has a special Laplace exponent ϕ is called a special subordinator and the corresponding subordinate Brownian motion is called a special subordinate Brownian motion.

For a function $f : \mathbb{R}^d \to \mathbb{C}$ we define $f^*(u) = \sup_{|x|\leqslant u} \operatorname{Re} f(x)$. The following lemma will play an important role in the sequel.

Lemma 3. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a negative definite function, then*

$$\frac{1}{2}\frac{s^2}{s^2+1}f^*(r) \leqslant f^*(sr) \leqslant 2(1+s^2)f^*(r), \qquad s, r > 0.$$

Proof. Since f is negative definite, $\operatorname{Re} f(x)$ and $f_r(x) = f(rx)$ are negative definite functions as well. The upper bound we get e.g. by using [19], (1.4) for $\operatorname{Re} f_r$. If $s \geqslant 1$, then we get the lower bound by monotonicity of f^*. For $s < 1$, by the upper bound

$$f^*(r) = f^*(rss^{-1}) \leqslant 2(1+s^{-2})f^*(rs),$$

which completes the proof. $\qquad \square$

Lemma 4. *Let $f : \mathbb{R}^d \to [0, \infty)$ and $\tilde{f}(u) = \sup_{|x|=u} f(x)$. Suppose that $\tilde{f}$ is positive on $(0, \infty)$ and $f(0) = 0$. If f satisfies $WLSC(\beta, \theta, C)$, then $WLSC\left(\beta, \theta, C^2\frac{\tilde{f}(\theta)}{f^*(\theta)}\right)$ holds for f^*, where $\frac{\tilde{f}(0)}{f^*(0)} = 1$.*

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Proof. We assume that $\theta > 0$, since the proof in the case $\theta = 0$ is similar. Note that, $\mathrm{WLSC}(\beta, \theta, C)$ holds for $\tilde{f}$. Hence,

$$\tilde{f}(u) \leqslant C^{-1}\tilde{f}(s), \qquad \theta \leqslant u \leqslant s. \tag{6}$$

Let $u \geqslant \theta$. Since $\tilde{f}$ is positive on $(0, \infty)$,

$$f^*(u) = \max\{f^*(\theta), \sup_{\theta \leqslant |x| \leqslant u} f(x)\} \leqslant \frac{f^*(\theta)}{\tilde{f}(\theta)} \sup_{\theta \leqslant |x| \leqslant u} f(x) \leqslant C^{-1}\frac{f^*(\theta)}{\tilde{f}(\theta)}\tilde{f}(u).$$

Hence, for $u \geqslant \theta$ and $\lambda > 1$, applying again $\mathrm{WLSC}(\beta, \theta, C)$ for $\tilde{f}$ we arrive at

$$f^*(\lambda u) \geqslant \tilde{f}(\lambda u) \geqslant C\lambda^\beta \tilde{f}(u) \geqslant C^2 \frac{\tilde{f}(\theta)}{f^*(\theta)} \lambda^\beta f^*(u).$$

$\square$

To the end of this section we assume that X_t is a Lévy process characterized by a triplet (A, ν, γ).

In the proof of following lemma we follow closely the ideas of [23], where authors proved similar result for isotropic stable processes.

Lemma 5. *Let $D \subset B_r$ and $x \in D \cap B_{r/2}$. Then there is a constant $C = C(d)$ such that*

$$P^x\left(|X_{\tau_D}| \geq r\right) = P_D(x, B_r^c) \leq Ch(r)E^x\tau_D,$$

where $h(r) = \|A\|r^{-2} + |\gamma + \int z\left(\mathbf{1}_{|z|<r} - \mathbf{1}_{|z|<1}\right)\nu(dz)|r^{-1} + \int_{\mathbb{R}^d} \min(1, |z|^2 r^{-2})\,\nu(dz)$.

Proof. For $f \in C_b^2(\mathbb{R}^d)$, by Dynkin formula we have

$$G_D(\mathcal{A}f)(x) = E^x f(X_{\tau_D}) - f(x), \quad x \in \mathbb{R}^d. \tag{7}$$

There is a non-decreasing function $g \in C_b^2([0, \infty))$ such that $g(s) = 0$, for $0 \leq s \leq 1/2$, $g(s) = 1$, for $s \geq 1$. Let $c_1 = \sup_s |g''(s)|$, then $\sup_s \frac{g'(s)}{s} \leqslant \frac{c_1}{2}$. We put $f(y) = g(|y|)$ and $f_r(y) = f(yr^{-1})$. Recall that $\mathrm{Tr}(A) \leqslant d\|A\|$. Hence

$$\sum_{j,k} A_{jk}\partial_{jk}^2 f(y) = \left(g''(|y|) - \frac{g'(|y|)}{|y|}\right)\frac{\langle y, Ay\rangle}{|y|^2} + \frac{g'(|y|)}{|y|}\mathrm{Tr}(A) \leqslant c_1\left(1 + \frac{d}{2}\right)\|A\|. \tag{8}$$

Since $g'(s) = 0$, for $s \geqslant 1$,

$$\langle z, \nabla f(y)\rangle = g'(|y|)\frac{\langle z, y\rangle}{|y|} \leqslant \frac{c_1}{2}|z|.$$

By (8), for $|z| < r$,

$$F_r(y, z) = f_r(y + z) - f_r(y) - \mathbf{1}_{|z|<r}\langle z, \nabla f_r(y)\rangle \leqslant \frac{|z|^2}{2}\sup_y \sum_{j,k}\partial_{jk}^2 f_r(y) \leqslant c_1\frac{d+2}{4}\frac{|z|^2}{r^2}.$$

And, for $|z| \geqslant r$, $F_r(x, y) \leqslant 1$. By (1) we have

$$\mathcal{A}f_r(y) = \int_{\mathbb{R}^d} F_r(y, z)\nu(dz) + \left\langle \gamma + \int_{\mathbb{R}^d}\left(\mathbf{1}_{|z|<r} - \mathbf{1}_{|z|<1}\right)z\nu(dz), \nabla f_r(y)\right\rangle + \sum_{j,k} A_{jk}\partial_{jk}^2 f_r(y)$$

$$\leqslant c_1\frac{d+2}{4}\int_{|z|<r}\left(1 \wedge \frac{|z|^2}{r^2}\right)\nu(dz) + c_1\frac{d+2}{2}\frac{\|A\|}{r^2} + \frac{c_1}{2}\frac{|\gamma + \int_{\mathbb{R}^d}\left(\mathbf{1}_{|z|<r} - \mathbf{1}_{|z|<1}\right)z\nu(dz)|}{r} \tag{9}$$

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Applying (7) to $f_r(y)$, we get

$$G_D(\mathcal{A}f_r)(x) = E^x f_r(X_{\tau_D}), \quad |x| \leq r/2. \tag{10}$$

Since $P^x\left(|X_{\tau_D}| \geq r\right) \leq E^x f_r(X_{\tau_D})$ and $G_D \mathbf{1} = E^x \tau_D$, the estimates (9) and (10) provide the conclusion. $\qquad\square$

Lemma 6. *For any $r \geqslant 0$,*

$$\frac{1}{8(1+2d)}\left(||A||r^2 + \int \left(1 \wedge (r|z|)^2\right)\nu(dz)\right) \leqslant \psi^*(r) \leqslant 2\left(||A||r^2 + \int \left(1 \wedge (r|z|)^2\right)\nu(dz)\right).$$

Proof. Let us observe that

$$\psi^*(r) \leqslant \left(\sup_{|z|\leqslant r}\langle z, Az\rangle + \sup_{|z|\leqslant r}\int(1-\cos\langle z,y\rangle)\nu(dy)\right) \leqslant 2\psi^*(r).$$

Since $\sup_{|z|\leqslant r}\langle z, Az\rangle = ||A||r^2$ it remains to prove

$$\frac{1}{4(1+2d)}\int_{\mathbb{R}^d}\min(1,(|z|r)^2)\,\nu(dz) \leqslant \sup_{|z|\leqslant r}\int(1-\cos\langle z,y\rangle)\nu(dy) \leqslant 2\int_{\mathbb{R}^d}\min(1,(|z|r)^2)\,\nu(dz). \tag{11}$$

Let $\tilde{\psi}(z) = \int(1-\cos\langle z,y\rangle)\nu(dy)$. Notice that (see e.g. [13], (5.4)),

$$\frac{|x|^2}{1+|x|^2} = \int_{\mathbb{R}^d}(1-\cos(\langle x,y\rangle))\,g(y)dy,$$

where

$$g(y) = \frac{1}{2}\int_0^\infty (2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}e^{-\frac{s}{2}}ds.$$

Hence, by the Fubini-Tonelli theorem

$$\begin{aligned}
\int_{\mathbb{R}^d}\min(1,(|z|r)^2)\,\nu(dz) &\leqslant 2\int_{\mathbb{R}^d}\frac{(|z|r)^2}{1+(|z|r)^2}\nu(dz) \\
&= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1-\cos(\langle zr,y\rangle))\,g(y)dy\nu(dz) = 2\int_{\mathbb{R}^d}\tilde{\psi}(yr)g(y)dy.
\end{aligned}$$

Since $\tilde{\psi}$ is a negative definite function, by Lemma 3 we have

$$\int_{\mathbb{R}^d}\min(1,(|z|r)^2)\,\nu(dz) \leqslant 4\sup_{|z|\leqslant r}\tilde{\psi}(z)\int_{\mathbb{R}^d}(1+|y|^2)g(y)dy = 4(1+2d)\sup_{|z|\leqslant r}\tilde{\psi}(z).$$

Since $1-\cos u = 2\sin^2\frac{u}{2} \leqslant 2\left(1\wedge|u|^2\right)$,

$$\int_{\mathbb{R}^d}\min(1,|z|^2|x|^2)\,\nu(dz) \geqslant \frac{1}{2}\tilde{\psi}(x).$$

Hence

$$\sup_{|x|\leqslant r}\tilde{\psi}(z) \leqslant 2\sup_{|x|\leqslant r}\int_{\mathbb{R}^d}\min(1,(|z||x|)^2)\,\nu(dz) = 2\int_{\mathbb{R}^d}\min(1,(|z|r)^2)\,\nu(dz),$$

which completes the proof of the inequality (11). $\qquad\square$

Since for symmetric processes $h(r) = \frac{\|A\|}{r^2} + \int \left(1 \wedge \frac{|z|^2}{r^2}\right) \nu(dz)$, we obtain the following corollary.

Corollary 7. *Let X_t be symmetric, then*

$$\frac{1}{2}\psi^*(r^{-1}) \leqslant h(r) \leqslant 8(1+2d)\psi^*(r^{-1}). \tag{12}$$

Remark 8. Instead of a direct calculation one can compare Pruitt's result [18] and [19], Remark 4.8 to obtain $\psi^*(r^{-1}) \approx h(r)$ under the assumption that there exists a constant c such that $|\operatorname{Im}\psi(x)| \leqslant c\operatorname{Re}\psi(x)$, $x \in \mathbb{R}^d$.

For subordinate Brownian motions easy calculations improve (12). Notice that ϕ is increasing.

Remark. Let X_t be a subordinate Brownian motion, then for $r > 0$,

$$\frac{1}{2}\phi(r^{-2}) \leqslant h(r) \leqslant (1+2d)\phi(r^{-2}).$$

Corollary 9. *Let X_t be symmetric. There exists a constant $C = C(d)$ such that, for $r > 0$, $s \leqslant r/2$,*

$$P^x(|X_{\tau_{B(0,s)}}| \geqslant r) \leqslant C\frac{\psi^*(r^{-1})}{\psi^*(s^{-1})}, \qquad |x| \leqslant s. \tag{13}$$

Moreover if ψ satisfies WLSC(β, θ, C^*), then, there exists a constant $C = C\left(d, \beta, C^*, \frac{\tilde{f}(\theta)}{f^*(\theta)}\right)$ such that, for $0 < r < \theta^{-1}$, $s \leqslant r/2$,

$$P^x(|X_{\tau_{B(0,s)}}| \geqslant r) \leqslant C\left(\frac{s}{r}\right)^\beta, \qquad |x| \leqslant s.$$

Proof. Since X_t is symmetric, by [19], Remark 4.8, and Lemma 3, we get

$$E^x\tau_{B_s} \leqslant E^x\tau_{B(x,2s)} \leqslant c_1\frac{1}{\psi^*(s^{-1})},$$

where $c_1 = c_1(d)$. Hence, the first claim follows by Lemma 5 and Corollary 7, while the second claim is a consequence of Lemma 4 and (13). We only have to check that $\tilde{\psi}(u) = \sup_{|x|=u}\psi(x)$ is positive on $(0,\infty)$. Suppose that there exists $u_0 > 0$, such that $\tilde{\psi}(u_0) = 0$. Then, by subadditivity of $\sqrt{\psi}$, we have that $\tilde{\psi}(nu_0) = 0$, for any $n \in \mathbb{N}$. Hence, by (6), $\tilde{\psi}(x) = 0$, for any $|x| \geqslant u_0 \vee \theta$. That implies that $\psi \equiv 0$, what we exclude. $\square$

3 Isotropic Unimodal Lévy Processes

In this section we assume that the process X_t is isotropic unimodal. In the first subsection we obtain estimates for the potential measure and capacity of balls, which are essential for the rest of the paper. Next, we use them to get estimates for a potential kernel and Green function of the ball. The next subsection contains some improvements of these estimates in the case subordinate Brownian motions. Subsection 3.3 is devoted to prove the Harnack inequality and regularity estimates for harmonic functions. In the last subsection we give some examples.

By ψ_0, ν_0 and G_0 we denote radial profiles of ψ, ν and G, respectively. For instance $\psi(x) = \psi_0(|x|)$.

Lemma 10. *([25]) Let X_t be a symmetric Lévy process, then the following conditions are equivalent:*

(1) X_t *is isotropic unimodal.*

(2) $G^\lambda(dx)$ *is isotropic unimodal, for $\lambda > 0$ (in the transient case for $\lambda \geqslant 0$), where $G^\lambda(dx) = \int_0^\infty e^{-\lambda t} P_t(dx) dt$ and $P_t(dx) = P^0(X_t \in dx)$.*

(3) $A = aI$, *for some $a \geqslant 0$ and ν is isotropic unimodal.*

Since X_t is isotropic its distribution is supported by the whole space, so it is transient for $d \geqslant 3$. Notice that $G(\{0\}) > 0$ iff ψ is bounded.

In the following proposition we prove that the characteristic exponent of an isotropic unimodal Lévy process is almost increasing.

Proposition 11. *We have, for any $x \in \mathbb{R}^d$,*

$$\psi^*(|x|) \leqslant 12\psi(x).$$

Proof. Let us define, for $r \geqslant 0$,

$$\tilde{\psi}_0(r) = 2 \int_0^\infty \big[1 - \cos(rz)\big]\nu_1(z)dz,$$

where $\nu_1(z) = \int_{\mathbb{R}^{d-1}} \nu_0(\sqrt{|w|^2 + z^2})dw$. Then, we have $\psi_0(r) = ar^2 + \tilde{\psi}_0(r)$, for some $a \geqslant 0$. Since ν_1 is non-increasing on $(0, \infty)$,

$$\begin{aligned}
\tilde{\psi}_0(r) &\geqslant \sum_{k=0}^\infty \int_{(\pi/3 + 2k\pi)/r}^{(5/3\pi + 2k\pi)/r} \nu_1(z)dz \geqslant \frac{4\pi}{3r} \sum_{k=0}^\infty \nu_1\left(\frac{5/3\pi + 2k\pi}{r}\right) \\
&\geqslant \frac{2}{3} \sum_{k=0}^\infty \int_{(5/3\pi + 2k\pi)/r}^{(5/3\pi + 2(k+1)\pi)/r} \nu_1(z)dz = \frac{2}{3}\nu_1\left[\frac{5\pi}{3r}, \infty\right).
\end{aligned}$$

We also note that $1 - \cos u \geqslant \frac{9}{2\pi^2}u^2$ if $|u| \leq \pi/3$. We have,

$$\begin{aligned}
\tilde{\psi}_0(r) &\geqslant \frac{9}{\pi^2} \int_0^{1/r} (zr)^2 \nu_1(dz) + 2\big[1 - \cos(1)\big]\nu_1\left(\left[\frac{1}{r}, \frac{5\pi}{3r}\right)\right) \\
&\geqslant \frac{9}{\pi^2} \left(\int_0^{1/r} (zr)^2 \nu_1(dz) + \nu_1\left(\left[\frac{1}{r}, \frac{5\pi}{3r}\right)\right)\right).
\end{aligned}$$

Hence,

$$\psi_0(r) \geqslant ar^2 + \frac{1}{3} \int_0^\infty \left(1 \wedge (zr)^2\right) \nu_1(z)dz.$$

Since the function $\int_0^\infty \left(1 \wedge (zr)^2\right) \nu_1(z)dz$ is non-decreasing and $1 - \cos u \leqslant 2(1 \wedge u^2)$,

$$\psi^*(r) = \sup_{|x| \leqslant r} \psi(x) \leqslant ar^2 + \sup_{s \leqslant r} 4 \int_0^\infty \left(1 \wedge (zs)^2\right) \nu_1(z)dz = ar^2 + 4 \int_0^\infty \left(1 \wedge (zr)^2\right) \nu_1(z)dz.$$

Finally, we get $\psi^*(r) \leqslant 12\psi_0(r)$. $\square$

3.1 Green function estimates

Lemma 12. *Assume that $d \geqslant 3$. Let $f(r) = G\left(B_{\sqrt{r}}\right)$. There exists a constant $C_1 = C_1(d)$ such that*

$$\frac{C_1}{\lambda \psi^*\left(\sqrt{\lambda}\right)} \leqslant \mathcal{L}f(\lambda) \leqslant \frac{36}{\lambda \psi^*\left(\sqrt{\lambda}\right)}, \qquad \lambda > 0.$$

Proof. Since

$$e^{-|z|^2} = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle z, x\rangle} e^{-|x|^2/4} dx, \quad z \in \mathbb{R}^d,$$

we have, for $\lambda > 0$,

$$Ee^{-\lambda|X_t|^2} = E\frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\sqrt{\lambda}\langle x, X_t\rangle} e^{-|x|^2/4} dx = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-t\psi(\sqrt{\lambda}x)} e^{-|x|^2/4} dx.$$

Integrating with respect to dt we have,

$$\lambda \mathcal{L}f(\lambda) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4} \frac{dx}{\psi(\sqrt{\lambda}x)}.$$

By Proposition 11, $\frac{1}{12}\psi^*(|x|) \leqslant \psi(x) \leqslant \psi^*(|x|)$. Hence,

$$\frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4} \frac{dx}{\psi^*(\sqrt{\lambda}|x|)} \leqslant \lambda \mathcal{L}f(\lambda) \leqslant \frac{12}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4} \frac{dx}{\psi^*(\sqrt{\lambda}|x|)}.$$

By Lemma 3,

$$\frac{1}{2\psi^*(\sqrt{\lambda})} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4}} \frac{1}{1+|x|^2} dx \quad \leqslant \quad \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4}} \frac{dx}{\psi^*(\sqrt{\lambda}|x|)} \quad \leqslant \quad \frac{2}{\psi^*(\sqrt{\lambda})} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4}} \frac{1+|x|^2}{|x|^2} dx.$$

The above estimates imply

$$\frac{C_1}{\psi^*(\sqrt{\lambda})} \leqslant \lambda \mathcal{L}f(\lambda) \leqslant \frac{36}{\psi^*(\sqrt{\lambda})},$$

where $C_1 = \frac{1}{2^{d+1}\pi^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4}} \frac{1}{1+|x|^2} dx$. $\qquad\square$

Proposition 13. *Assume that $d \geqslant 3$. There is a constant $C_2 = C_2(d)$ such that*

$$\frac{C_2}{\psi^*(r^{-1})} \leqslant G\left(B_r\right) \leqslant \frac{36e}{\psi^*(r^{-1})}, \qquad r > 0.$$

Proof. Let $f(r) = G\left(B_{\sqrt{r}}\right)$. Since $f(u)$ is non-decreasing

$$f(u) \leqslant \frac{e}{u} \int_u^\infty e^{-s/u} f(s) ds \leqslant \frac{e}{u} \mathcal{L}(f)(u^{-1}).$$

By Lemma 12 we get

$$G\left(B_r\right) \leqslant \frac{36e}{\psi^*(r^{-1})}. \tag{14}$$

By Lemma 3 we have, for $u \leqslant s$, $\frac{\psi^*(u^{-1/2})}{\psi^*(s^{-1/2})} \leqslant 4\frac{s}{u}$. Lemma 12 and (14) give us, for $\kappa > 1$,

$$\begin{aligned}
\int_{\kappa u}^\infty e^{-s/u} f(s) ds &\leqslant 36e \int_{\kappa u}^\infty e^{-s/u} \frac{ds}{\psi^*(s^{-1/2})} \leqslant \frac{144e}{u\psi^*(u^{-1/2})} \int_{\kappa u}^\infty e^{-s/u} s\, ds \\
&= \frac{144e\Gamma(2, \kappa)u}{\psi^*(u^{-1/2})} \leqslant c_1 \Gamma(2, \kappa) \mathcal{L}(f)(u^{-1}),
\end{aligned}$$

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where $c_1 = \frac{144e}{C_1}$. Hence, for κ such that $c_1 \Gamma(2, \kappa) = \frac{1}{2}$ we have

$$\mathcal{L}(f)(u^{-1}) \leqslant 2 \int_0^{\kappa u} e^{-s/u} f(s) ds \leqslant 2u f(\kappa u).$$

Again, by Lemmas 12 and 3 we infer

$$G(B_r) \geqslant \frac{C_1}{2\psi^*(\sqrt{\kappa} r^{-1})} \geqslant \frac{C_1}{4(\kappa + 1)\psi^*(r^{-1})}.$$

$\square$

By Cap^λ, $\lambda \geqslant 0$, we denote the λ-capacity with respect to X_t. When $\lambda = 0$ we omit a superscript "0". For any non-empty compact set $A \subset \mathbb{R}^d$ there exists a measure ρ_A (see e.g. [3]), called the equilibrium measure, such that is supported by A and

$$G\rho_A(F) = \int G(F - x)\rho_A(dx) = \int_F P^x(T_A < \infty)dx, \quad F \subset \mathbb{R}^d. \tag{15}$$

Moreover $\rho_A(A) = \mathrm{Cap}(A)$. If the potential measure is absolutely continuous, then

$$G\rho_A(x) = P^x(T_A < \infty), \qquad x \in \mathbb{R}^d.$$

Proposition 14. *Let $d \geqslant 3$. There exists a constant $C = C(d)$ such that, for any $r \geqslant 0$,*

$$C^{-1}\psi^*(r^{-1})r^d \leqslant \mathrm{Cap}\left(\overline{B_r}\right) \leqslant C\psi^*(r^{-1})r^d.$$

Proof. Since $d \geqslant 3$, $\mathrm{Cap}(\{0\}) = 0$, so we may assume that $r > 0$. By Lemma 10, G is radially non-increasing. Let $x \in \overline{B_r}$, then

$$G\left(\overline{B_r} - x\right) = G(\{0\}) + \int_{B_r} G(y - x)dy.$$

By [21], Proposition 5.3, there exists a constant $c_1 = c_1(d)$ such that

$$c_1 \int_{B_r} G(y)dy \leqslant \int_{B_r} G(y - x)dy \leqslant \int_{B_r} G(y)dy.$$

Hence

$$c_1 G\left(\overline{B_r}\right) \leqslant G\left(\overline{B_r} - x\right) \leqslant G\left(\overline{B_r}\right).$$

By (15),

$$|\overline{B_r}| = G\rho_{\overline{B_r}}\left(\overline{B_r}\right).$$

This preparation yields

$$c_1 |\overline{B_r}| \leqslant G\left(\overline{B_r}\right) \mathrm{Cap}\left(\overline{B_r}\right) \leqslant |\overline{B_r}|,$$

which combined with Proposition 13 implies

$$\frac{c_1 |B_1|}{36e}\psi^*(r^{-1})r^d \leqslant \mathrm{Cap}\left(\overline{B_r}\right) \leqslant \frac{|B_1|}{C_2}\psi^*(r^{-1})r^d.$$

$\square$

Remark. Similar calculations provide the following estimates for λ-resolvent measure and λ-capacity

$$G^\lambda(B_r) \approx \frac{1}{\lambda + \psi^*(r^{-1})} \text{ and } \mathrm{Cap}^\lambda(\overline{B_r}) \approx r^d \left(\lambda + \psi^*(r^{-1})\right).$$

By [25], Theorem 1, it is known that the smallest capacity among sets with the same volume is attained for a closed ball. Therefore, by the above proposition and Lemma 3 we obtain the following result.

Corollary 15. *There exists a constant $C_3 = C_3(d)$ such that, for any non-empty Borel set A,*

$$\mathrm{Cap}(A) \geqslant C_3 \psi^*(|A|^{-1/d})|A|. \tag{16}$$

To our best knowledge the following upper bound for the potential kernel was known only for subordinate Brownian motions and the lower one for special subordinate Brownian motions (see e.g. [26]). They were obtained as a consequence of appropriate estimates for the potential measure and the potential kernel of the subordinator, respectively.

Theorem 16. *Let $d \geqslant 3$. Then there exists a constant $C_4 = C_4(d)$ such that*

$$G(x) \leqslant \frac{C_4}{|x|^d \psi^*(|x|^{-1})}, \qquad x \in \mathbb{R}^d.$$

If additionally ψ satisfies $WLSC(\beta, R^{-1}, C^*)$, then

$$\frac{C_5}{|x|^d \psi^*(|x|^{-1})} \leqslant G(x), \qquad |x| \leqslant bR,$$

where $b = c(d, \beta)(C^*)^{1/\beta} < 1$ and $C_5 = c(d)b^d$.

Proof. Since G is radially non-increasing,

$$\int_{B(0,|x|)} G(y)dy \geqslant |B_1||x|^d G(x).$$

By Proposition 13

$$|x|^d G(x) \leqslant \frac{36e}{|B_1|} \frac{1}{\psi^*(|x|^{-1})},$$

which completes the proof of the upper bound.

Again, by radial monotonicity, we have, for $\kappa > 1$,

$$G(x) \geqslant \frac{G\left(B_{\kappa|x|} \setminus B_{|x|}\right)}{|B_{\kappa|x|} \setminus B_{|x|}|} \geqslant \frac{1}{|B_1|\kappa^d} \frac{G\left(B_{\kappa|x|} \setminus B_{|x|}\right)}{|x|^d}.$$

Suppose that ψ satisfies $WLSC(\beta, R^{-1}, C^*)$, then by Propositions 13 and 11, for $\kappa|x| \leqslant R$,

$$\begin{aligned} G\left(B_{\kappa|x|} \setminus B_{|x|}\right) &= G\left(B_{\kappa|x|}\right) - G\left(B_{|x|}\right) \geqslant \frac{C_2}{\psi^*\left((\kappa|x|)^{-1}\right)} - \frac{36e}{\psi^*(|x|^{-1})} \\ &= \frac{36e}{\psi^*(|x|^{-1})}\left(c_1 \frac{\psi^*(|x|^{-1})}{\psi^*((\kappa|x|)^{-1})} - 1\right) \geqslant \frac{36e}{\psi^*(|x|^{-1})}\left(\frac{c_1 C^*}{12}\kappa^\beta - 1\right). \end{aligned}$$

Hence, for $\kappa = (24/(c_1 C^*))^{\frac{1}{\beta}}$, we get

$$G(x) \geqslant \frac{36e}{|B_1|\kappa^d} \frac{1}{|x|^d \psi^*(|x|^{-1})}, \qquad |x| \leqslant R/\kappa.$$

$\square$

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Corollary 17. *Let $d \geqslant 3$. If $WLSC(\beta, R^{-1}, C)$ holds for ψ, then there exists a constant $b < 1$ such that*

$$G(x) \approx G(B_{|x|})|x|^{-d}, \qquad |x| \leqslant bR.$$

The above comparability is crucial in our proof of the scale invariant Harnack inequality. In the next subsection we show the converse of Corollary 17 in the case of special subordinate Brownian motions.

Remark 18. If ψ satisfies $WLSC(\beta, R^{-1}, C^*)$, then a local doubling condition for G holds. This means there exists constant $C = C(d, \beta, C^*)$ such that $CG(x) \leqslant G(2x) \leqslant G(x)$, for $0 < |x| \leqslant bR/2$, where a constant b is from Theorem 16.

Standard arguments provide the following proposition.

Proposition 19. *Let $d \geqslant 3$. Suppose that ψ satisfies $WLSC(\beta, R^{-1}, C^*)$, then, for any $\varepsilon \in (0,1)$, there exists a constant $L = L(\varepsilon, d, \beta, C^*) > 1$ such that, for $r \leqslant R$,*

$$G_{B_r}(x, y) \geqslant \varepsilon G(x - y), \qquad L|x - y| \leqslant (r - |x|) \vee (r - |y|).$$

Proof. Since ψ satisfies WLSC, it is unbounded. Therefore $G(\{0\}) = 0$ and due to Lemma 10 the potential measure is absolutely continuous. We may and do assume that $|y| \leqslant |x|$ and $L \geqslant b^{-1}$. Since $|X_{\tau_{B_r}} - y| \geqslant r - |y| \geqslant L|x - y|$, by radial monotonicity of G and (2),

$$G_{B_r}(x, y) = G(x - y) - E^x G(X_{\tau_{B_r}} - y) \geqslant G(x - y) - G(L(x - y)).$$

By this, Theorem 16 and Lemma 3,

$$G_{B_r}(x, y) \;\geqslant\; G(x - y)\left(1 - \frac{C_4}{C_5 L^d}\frac{\psi^*(|x - y|^{-1})}{\psi^*((L|x - y|)^{-1})}\right) \geqslant G(x - y)\left(1 - \frac{4C_4}{C_5 L^{d-2}}\right).$$

Hence, for $L = \left(\frac{4C_4}{C_5(1-\varepsilon)}\right)^{1/(d-2)} \vee b^{-1}$ we obtain

$$G_{B_r}(x, y) \geqslant \varepsilon G(x - y).$$

$\square$

3.2 Subordinate Brownian motions

In this subsection we improve Theorem 16 and Proposition 19 in the case of subordinate Brownian motions. Namely, we prove that $b = 1$ and $L = 2$, for some $\varepsilon > 0$. We assume in this subsection that X_t is a subordinate Brownian motion.

The following lemma is well known (see e.g. [10]), but for the convenience of the reader we prove it with a short and simple proof.

Proposition 20. *For $r > 0$*

$$\frac{1 - 2e^{-1}}{2\phi(r^{-1})} \leqslant U[0, r) \leqslant \frac{e}{\phi(r^{-1})}.$$

Proof. Notice that for $\lambda > 1$, $\phi(\lambda r) \leqslant \lambda \phi(r)$. Hence,

$$\frac{1}{2\phi((2r)^{-1})} \leqslant \frac{1}{\phi(r^{-1})} = \int_0^{2r} e^{-r^{-1}t}U(dt) + \int_{2r}^\infty e^{-r^{-1}t}U(dt)$$

$$\leqslant U[0,2r) + e^{-1}\int_{2r}^\infty e^{-(2r)^{-1}t}U(dt) \leqslant U[0,2r) + \frac{e^{-1}}{\phi((2r)^{-1})},$$

which proves the lower bound.

On the other hand

$$U[0,r) \leqslant e \int_0^r e^{-r^{-1}t}U(dt) \leqslant \frac{e}{\phi(r^{-1})}.$$

$\square$

The following theorem is an improvement of Theorem 16. Such result is known (see e.g. [26], Theorem 1), under an additional assumption that ϕ is a special Bernstein function.

Theorem 21. *Let $d \geqslant 3$ and X_t be a subordinate Brownian motion. If ϕ satisfies $WLSC(\beta, R^{-2}, C^*)$, then there exists a constant $C_6 = C_6(d, \beta, C^*)$ such that*

$$G(x) \geqslant \frac{C_6}{|x|^d \phi(|x|^{-2})}, \qquad |x| \leqslant R.$$

Proof. Let $\kappa < 1$. By (5) we have

$$G(x) \geqslant \int_{\kappa|x|^2}^{|x|^2} g_t(x)U(dt) \geqslant (g_\kappa(\mathbf{1}) \wedge g_1(\mathbf{1})) |x|^{-d}U[\kappa|x|^2, |x|^2),$$

where $\mathbf{1} = (1, 0, \ldots, 0)$. Suppose that ϕ satisfies $WLSC(\beta, R^{-2}, C^*)$, then by Lemma 20, for $|x| \leqslant R$,

$$U[\kappa|x|^2, |x|^2) = U[0, |x|^2) - U[0, \kappa|x|^2) \geqslant \frac{1 - 2e^{-1}}{2\phi(|x|^{-2})} - \frac{e}{\phi((\kappa|x|^2)^{-1})}$$

$$= \frac{1 - 2e^{-1}}{2\phi(|x|^{-2})}\left(1 - c_1 \frac{\phi(|x|^{-2})}{\phi((\kappa|x|^2)^{-1})}\right) \geqslant \frac{1 - 2e^{-1}}{2\phi(|x|^{-2})}\left(1 - c_2 \kappa^\beta\right),$$

where $c_2 = \frac{2e^2}{(e-2)C^*}$. Hence, for $\kappa = (2c_2)^{-\frac{1}{\beta}}$, we get

$$G(x) \geqslant \frac{c_3}{|x|^d \phi(|x|^{-2})}, \qquad |x| \leqslant R,$$

where $c_3 = \frac{1 - 2e^{-1}}{4}(g_\kappa(\mathbf{1}) \wedge g_1(\mathbf{1}))$.

$\square$

The following theorem is a converse of the above theorem (and Corollary 17) in the case of special subordinate Brownian motions. Since the comparability in Corollary 17 is the key ingredient in the proof of the Krylov-Safonov estimate it seems that the approach of Bass and Levin for proving the Harnack inequality can not be used if ϕ does not satisfy WLSC.

Theorem 22. *Let $d \geqslant 3$ and X_t be a special subordinate Brownian motion. There exists a constant C such that $G(x) \geqslant \frac{C}{|x|^d \phi(|x|^{-2})}$, for $|x| \leqslant R$ iff ϕ satisfies $WLSC(\beta, R^{-2}, 1)$, for some $\beta > 0$.*

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Proof. Due to Theorem 21 it is enough to show that the existence of a constant c_1 such that

$$G(x) \geqslant \frac{c_1}{|x|^d \phi(|x|^{-2})}, \qquad |x| \leqslant R, \tag{17}$$

implies the weak lower scaling condition for ϕ. Suppose that (17) holds. Since the process is transient

$$\int_{B_R} G(x) dx \leqslant G(B_R) < \infty,$$

which combined with (17) shows that ϕ is unbounded and consequently the potential measure of X_t is absolutely continuous. Since ϕ is a special Bernstein function, by (5), we have

$$G(x) = \int_0^\infty g_s(x) u(s) ds,$$

where u is non-increasing. By (4),

$$\left| \left(\frac{1}{\phi} \right)' \right| (\lambda) = \int_0^\infty s e^{-\lambda s} u(s) ds \geqslant u(\lambda^{-1}) \int_0^{\lambda^{-1}} s e^{-\lambda s} ds = (1 - 2e^{-1}) \lambda^{-2} u(\lambda^{-1}). \tag{18}$$

Since $\frac{1}{\phi}$ is completely monotone $\left| \left(\frac{1}{\phi} \right)' \right|$ is non-increasing. Due to (18), monotonicity of $\left| \left(\frac{1}{\phi} \right)' \right|$ and u implies

$$u(s) \leqslant \frac{1}{1 - 2e^{-1}} \left| \left(\frac{1}{\phi} \right)' (|x|^{-2}) \right| \left(|x|^{-4} \vee s^{-2} \right).$$

Hence,

$$G(x) \leqslant \frac{1}{1 - 2e^{-1}} \left| \left(\frac{1}{\phi} \right)' (|x|^{-2}) \right| \int_0^\infty \left(|x|^{-4} \vee s^{-2} \right) g_s(x) ds = c_2 \frac{\left(-\frac{1}{\phi} \right)' (|x|^{-2})}{|x|^{d+2}},$$

where $c_2 = \frac{16\Gamma\left(d/2+1,\frac{1}{4}\right) + \gamma\left(d/2-1,\frac{1}{4}\right)}{4\pi^{d/2}(1-2e^{-1})}$. Using (17),

$$\frac{1}{\phi}(\lambda) \leqslant \frac{c_2}{c_1} \left(-\frac{1}{\phi} \right)' (\lambda) \lambda, \quad \text{for } \lambda \geqslant R^{-2}.$$

That is, for $\lambda \geqslant R^{-2}$, $\phi(\lambda) \leqslant \frac{c_2}{c_1} \phi'(\lambda) \lambda$. Let $\beta < \frac{c_1}{c_2}$, then the function $\phi(u) u^{-\beta}$ is increasing on $[R^{-2}, \infty)$, since $\left(\phi(u) u^{-\beta} \right)' > 0$. In consequence

$$\frac{\phi(\lambda u)(\lambda u)^{-\beta}}{\phi(u) u^{-\beta}} \geqslant 1, \qquad \lambda \geqslant 1, u \geqslant R^{-2},$$

which completes the proof. $\qquad \square$

Let D be an open set and $x \in D$. Denote $\delta_D(x)$ a distance x from a boundary of D. The following theorem improves Proposition 19. Like in the case of Theorem 21 such result was known only for special subordinate Brownian motions for which the characteristic exponent or its derivative satisfies some scaling conditions (see e.g. [15], [14]). These results were obtained by standard arguments we used in Proposition 19, therefore the appropriate constants depend on a process. Our proof for a special subordinate Brownian motion does not require any scaling properties and the appearing constant depends only on the dimension.

Theorem 23. *Let $d \geqslant 3$ and D be an open set. Suppose that ϕ is a unbounded special Bernstein function, then*

$$G_D(x,y) \geqslant C_7 G(x-y), \qquad 2|x-y| \leqslant \delta_D(x) \vee \delta_D(y). \tag{19}$$

where $C_7 = \frac{\Gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)}{\Gamma\left(\frac{d}{2}-1\right)} \left(1 - e^{-\frac{3}{4}}\right)$.

If ϕ is only a Bernstein function but satisfies $WLSC(\beta, R^{-2}, C^*)$, then (19) holds, if additionally $|x-y| \leqslant R$ with a constant $C = C(d, \beta, C^*)$ instead of C_7.

Proof. Let us assume that $\delta_D(x) \leqslant \delta_D(y)$ and $x \neq y$. Since $2|x-y| \leqslant \delta_D(y) \leqslant |X_{\tau_D} - y|$, by (2) and radial monotonicity,

$$G_D(x,y) = G(x,y) - E^x G(X_{\tau_D} - y) \geqslant G(x-y) - G(2(x-y)). \tag{20}$$

Let us define a function

$$K_p(q) = \int_0^\infty (4\pi s)^{-d/2} \left(e^{-\frac{p^2}{4s}} - e^{-\frac{q^2}{4s}}\right) U(ds), \qquad p, q \geqslant 0.$$

Due to (5) we have

$$G(x-y) - G(2(x-y)) = K_{|x-y|}(2|x-y|). \tag{21}$$

Moreover,

$$
\begin{aligned}
K_{|x-y|}(2|x-y|) &\geqslant \int_0^{|x-y|^2} (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} \left(1 - e^{-\frac{3|x-y|^2}{4s}}\right) U(ds) \\
&\geqslant \left(1 - e^{-\frac{3}{4}}\right) \int_0^{|x-y|^2} (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} U(ds).
\end{aligned}
\tag{22}
$$

Suppose that ϕ is a special Bernstein function. Then there exists a non-increasing function u such that $U(ds) = u(s)ds$. Monotonicity of u yields

$$
\begin{aligned}
\int_{|x-y|^2}^\infty (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} u(s)ds &\leqslant \frac{\gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)}{4\pi^{d/2}} u(|x-y|^2)|x-y|^{2-d} \\
&\leqslant \frac{\gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)}{\Gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)} \int_0^{|x-y|^2} (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4u}} u(s)ds.
\end{aligned}
$$

Hence,

$$G(x-y) \leqslant \left(1 + \frac{\gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)}{\Gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)}\right) \int_0^{|x-y|^2} (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} u(s)ds \leqslant c_2 K_{|x-y|}(2|x-y|),$$

where $c_2 = \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-1,\frac{1}{4}\right)\left(1-e^{-\frac{3}{4}}\right)}$. In consequence

$$G_D(x,y) \geqslant c_2^{-1} G(x-y), \qquad 2|x-y| \leqslant \delta_D(y).$$

Now, let us suppose that ϕ is only Bernstein function (it is no longer assumed that it is special) satisfying $WLSC(\beta, R^{-2}, C^*)$. Then, by the proof of Theorem 21 and Theorem 16 there exists a constant $\kappa < 1$, such that

$$\int_{\kappa|x-y|^2}^{|x-y|^2} g_s(x-y)U(ds) \geqslant \frac{C_6}{C_4} G(x-y), \qquad |x-y| \leqslant R.$$

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Hence, by (20)–(22)

$$G_D(x,y) \geqslant \left(1 - e^{-\frac{3}{4}}\right) \frac{C_6}{C_4} G(x-y).$$

$\square$

Remark. Let $d \geqslant 3$. Suppose that ϕ is an unbounded special Bernstein function then there exists a constant $C = C(d)$ such that, for any $r > 0$

$$CG(x-y) \leqslant G_{B_r}(x,y) \leqslant G(x-y), \qquad x, y \in B(0, r/5).$$

If ϕ is only a Bernstein function satisfying $\mathrm{WLSC}(\beta, R^{-2}, C^*)$, then there exists a constant $C = C(d, \beta, C^*)$ such that, for any $r \leqslant R$ the above inequality holds.

3.3 Harnack inequality and Hölder regularity

The goal of this subsection is to prove the main results of this paper, that is Theorems 1 and 2. In this subsection we assume that X_t is an isotropic unimodal Lévy process with the characteristic exponent satisfying $\mathrm{WLSC}(\beta, \theta, C^*)$. Since ψ satisfies the weak lower scaling condition, therefore is unbounded. Hence, the potential measure is absolutely continuous. Let $R = \theta^{-1}$. By L we denote the constant from Proposition 19 for $\varepsilon = C_7$ or $L = 2$ in the case of special subordinate Brownian motions and let $r_0 = \frac{r}{2L+1}$.

In the proof of the following proposition we follow closely the ideas of [4], where symmetric stable Lévy processes were considered.

Proposition 24. *Let $d \geqslant 3$ and ψ satisfy $\mathrm{WLSC}(\beta, R^{-1}, C^*)$. Then there exists a constant $C_8 = C(d, \beta, C^*)$ such that, for any $r \leqslant R$, and any non-negative function H such that $\mathrm{supp}\, H \subset \overline{B_r}^c$,*

$$E^x H(X_{\tau_{B_{r_0}}}) \leqslant C_8 E^y H(X_{\tau_{B_r}}), \qquad x, y \in B_{\frac{r_0}{2}}.$$

Proof. Due to Lemma 10, the Ikeda-Watanabe formula (3) and (2) we obtain that the Poisson kernel of B_r exists and

$$E^y H(X_{\tau_{B_r}}) = \int_{B_r^c} H(z) P_{B_r}(y, z) dz,$$

where

$$P_{B_r}(y, z) = \int_{B_r} G_{B_r}(y, w) \nu(z - w) dw.$$

Hence, it is enough to prove there is a constant c_1 such that

$$P_{B_{r_0}}(x, z) \leqslant c_1 P_{B_r}(y, z), \qquad x, y \in B_{\frac{r_0}{2}}, |z| > r. \tag{23}$$

By Proposition 19 and radial monotonicity of G,

$$P_{B_r}(y, z) \geqslant \int_{B_{r_0}} G_{B_r}(y, w) \nu(z - w) dw \geqslant C_7 G_0(2r_0) \int_{B_{r_0}} \nu(z - w) dw. \tag{24}$$

Since ν is radially non-increasing, for $w \in B_{\frac{3}{4}r_0}$,

$$\nu(w - z) \leqslant \frac{\int_{B\left(w_z, \frac{r_0}{8}\right)} \nu(u - z) du}{|B_{\frac{r_0}{8}}|} \leqslant c_2 r_0^{-d} \int_{B_{r_0}} \nu(u - z) du,$$

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where $w_z = w + \frac{r(z-w)}{8|z-w|}$ and $c_2 = \frac{8^d}{|B_1|}$. Hence, by a doubling condition (see Remark 18) and radial monotonicity of G,

$$
\begin{aligned}
P_{B_{r_0}}(x,z) \;\leqslant\;& \int_{B_{r_0}} G(x-w)\nu(w-z)\,dw \\
\leqslant\;& G_0\left(\frac{r_0}{4}\right)\int_{B_{r_0}\setminus B_{\frac{3}{4}r_0}} \nu(w-z)\,dw + c_2 r_0^{-d}\int_{B_{r_0}}\nu(u-z)\,du\int_{B_{\frac{3}{4}r_0}} G(x-w)\,dw \\
\leqslant\;& \left(c_3 G_0(2r_0) + c_2 r_0^{-d}\int_{B\left(x,\frac{5}{4}r_0\right)} G(x-w)\,dw\right)\int_{B_{r_0}}\nu(u-z)\,du \\
\leqslant\;& c_4\left(G_0(2r_0) + (2r_0)^{-d}G(B_{2r_0})\right)\int_{B_{r_0}}\nu(u-z)\,du,
\end{aligned}
$$

where $c_4 = c_3 \vee (2^d c_2)$. Corollary 17 provides

$$
P_{B_{r_0}}(x,z) \leqslant c_5 G_0(2r_0)\int_{B_{r_0}}\nu(u-z)\,du,
$$

for some constant $c_5 = c_5(d,\beta,C^*)$. Due to (24) this implies (23), which completes the proof. $\quad\square$

In the following proposition we prove the Krylov-Safonov estimate, which is the crucial for proving the scale invariant Harnack inequality. In the proof we use some ideas of [21], Lemma 6.2.

Proposition 25. *Let $d \geqslant 3$ and ψ satisfy $WLSC(\beta, R^{-1}, C^*)$. There exists a constant $C_9 = C(d,\beta,C^*)$ such that for any $r \leqslant R$ and any compact $A \subset B_{r_0}$,*

$$
P^x(T_A < \tau_{B_r}) \geqslant C_9 \frac{|A|}{|B_{r_0}|}, \qquad x \in B_{r_0}.
$$

Proof. Let B be an open set and $A \subset B$ be compact. Similarly to (15), let

$$
G_B \rho_A(x) = \int_A G_B(x,y)\rho_A(dy).
$$

Then, by the strong Markov property and (2),

$$
\begin{aligned}
G_B \rho_A(x) \;=\;& G\rho_A(x) - E^x G\rho_A(X_{\tau_B}) = P^x(T_A < \infty) - E^x P^{X_{\tau_B}}(T_A < \infty) \\
\leqslant\;& P^x(T_A < \infty) - E^x[P^{X_{\tau_B}}(T_A < \infty), T_A > \tau_B] = P^x(T_A < \tau_B).
\end{aligned}
$$

On the other hand

$$
G_B \rho_A(x) \;\geqslant\; \inf_{y\in A} G_B(x,y)\rho_A(A).
$$

This implies

$$
P^x(T_A < \tau_B) \geqslant \inf_{z\in A} G_B(x,z)\,\mathrm{Cap}(A). \tag{25}
$$

By Proposition 19 and radial monotonicity of G, for $x \in B_{r_0}$,

$$
\inf_{z\in B_{r_0}} G_{B_r}(x,z) \geqslant C_7 \inf_{z\in B_{r_0}} G(x-z) \geqslant C_7 G_0(2r_0).
$$

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By Theorem 16, $G_0(2r_0) \geqslant \frac{C_5}{(2r_0)^d \psi^*((2r_0)^{-1})}$. Combining this with (25) for $A \subset B = B_r$, and (16) we obtain

$$P^x(T_A < \tau_{B_r}) \geqslant c_1 \frac{|A| \psi^*(|A|^{-1/d})}{r_0^d \psi^*((2r_0)^{-1})},$$

where $c_1 = \frac{C_3 C_5 C_7}{2^d}$. By Lemma 3, for $A \subset B_{r_0}$, there exists a constant $c_2 = c_2(d)$ such that $\frac{\psi^*(|A|^{-1/d})}{\psi^*((2r_0)^{-1})} \geqslant c_2$. Hence,

$$P^x(T_A < \tau_{B_r}) \geqslant c_1 c_2 |B_1| \frac{|A|}{|B_{r_0}|}.$$

$\square$

Let us notice that until now, under the assumption that X_t is isotropic unimodal with its characteristic exponent satisfying WLSC(β, R^{-1}, C^*), all the constants that appear in the paper depend only on d, β, C^*. None of them depends on R or θ, respectively.

Now, we are ready to prove the main results of our paper.

Proof of Theorem 1. We prove the result for bounded harmonic functions. The boundedness assumption can be removed in a similar way as in [22], Theorem 2.4. Assume that ψ satisfy WLSC(β, θ, C^*). Let $R_0 > 0$. We prove that there exists a constant $c_1 = c_1(R_0)$ such that, for any function h non-negative on $\mathbb{R}^d$ and harmonic in a ball B_r, $r \leqslant R_0$,

$$\sup_{x \in B(0, r/2)} h(x) \leqslant C \inf_{x \in B(0, r/2)} h(x). \tag{26}$$

Recall that $R = \theta^{-1}$. With Propositions 24 and 25 at hand we can use the approach of Bass and Levin ([2]) to get the existence of constants $c_2 = c_2(d, \beta, C^*)$ and $a = a(d, \beta, C^*) < 1$ such that, for any function h non-negative and bounded on $\mathbb{R}^d$ and harmonic in a ball B_r, $r \leqslant R$,

$$\sup_{x \in B_{ar}} h(x) \leqslant c_2 \inf_{x \in B_{ar}} h(x).$$

Next, we use the standard chain argument to get

$$\sup_{x \in B_{r/2}} h(x) \leqslant c_3 \inf_{x \in B_{r/2}} h(x),$$

where $c_3 = c_3(d, c_2, a)$. If $R_0 \leqslant R$ we have (26). Notice, that if ψ satisfies the global weak lower scaling condition ($R = \infty$) we get the global scale invariant Harnack inequality, since we can take $c_1 = c_3$ and c_3 does not depend on R_0. For $R_0 > R$, one can use again the chain argument to get (26), for any harmonic function on B_r, $r \leqslant R_0$. But then the constant $c_1 = c_1\left(d, c_3, \frac{R_0}{R}\right)$. $\square$

To deal with dimension $d \leq 2$ we use the idea from [17], which relies on extending harmonic functions to higher dimensional spaces.

Corollary 26. *Let $d \leqslant 2$. Suppose that there exists an unimodal isotropic Lévy process $Y_t \in \mathbb{R}^3$, such that X_t is a projection of Y_t. If ψ satisfy WLSC(β, θ, C^*), then the scale invariant Harnack inequality holds.*

Proof. We present only the one-dimensional case. Without loss of generality we can assume that $X_t = Y_t^{(1)}$, where $Y_t = (Y_t^{(1)}, Y_t^{(2)}, Y_t^{(3)})$. Suppose that h is harmonic and non-negative with respect to X_t in $(-r, r)$, then by the strong Markov property a function $f : \mathbb{R}^3 \to [0, \infty)$ defined by $f(x^{(1)}, x^{(2)}, x^{(3)}) = h(x^{(1)})$ is harmonic with respect to Y_t in $(-r, r) \times \mathbb{R}^2$. Since X_t is isotropic we obtain that the characteristic exponent of the process Y_t satisfies WLSC(β, R^{-1}, C^*). Due to Theorem 1 the scale invariant Harnack inequality holds for Y_t, so it must hold for X_t. $\square$

Proof of Theorem 2. With the Krylov-Safonov type estimate (Proposition 25) and the second part of Corollary 9 the proof is similar to the proof in [2], Theorem 4.1, therefore it is omitted. $\square$

Corollary 27. *Let $d \leqslant 2$. Suppose that there exists an unimodal isotropic Lévy process $Y_t \in \mathbb{R}^3$, such that X_t is a projection of Y_t. If ψ satisfy $WLSC(\beta, \theta, C^*)$, then the conclusion of Theorem 2 holds for X_t.*

Let us remark that in general we can not find an isotropic Lévy process Y_t in higher dimension such that X_t is a projection of Y_t. If there exists such process Y_t, then $\nu_0(|x|) = \int_{\mathbb{R}^{3-d}} \nu_0^Y(\sqrt{|x|^2 + |y|^2}) dy$. Hence ν_0 must be continuous on $(0, \infty)$. Therefore if ν_0 is not continuous the construction of Y_t is impossible.

On the other hand any Bernstein function defines a subordinate Brownian motion in every dimension hence the following theorem holds with no restriction on dimension.

Theorem 28. *Let $d \geqslant 1$ and X_t be a subordinate Brownian motion. Suppose that ϕ satisfies $WLSC(\beta, \theta, C^*)$. Then the scale invariant Harnack inequality as well as the conclusion of Theorem 2 hold. Moreover, if ϕ satisfies the global weak lower scaling condition, then the global scale invariant Harnack inequality holds.*

3.4 Examples

Recall that $\nu(dx) = \nu_0(|x|)dx$. If we do not mention $d \geqslant 3$.

Example 1. Let $A = 0$ and $\nu_0(r) = \frac{f(r)}{r^d}$, $r \in (0, 1)$, where $f(r)$ is non-increasing and non-negative. If $f(\lambda r) \leqslant c\lambda^{-\beta} f(r)$, for $\lambda > 1$ and $\lambda r \leqslant 1$ then the scale invariant Harnack inequality holds, since ψ satisfies $WLSC(\beta, \theta, C)$ for some $\theta \geqslant 0$ and $C > 0$.

Example 2. Let $A = 0$ and $\nu_0(r) = \frac{f(r)}{r^d}$, $r \in (0, \infty)$, where $f(r)$ is non-increasing and non-negative. If $f(\lambda r) \leqslant c\lambda^{-\beta} f(r)$, for $r > 0$ and $\lambda > 1$ then the global scale invariant Harnack inequality holds. For instance this example is applicable for the following processes, $(\alpha, \alpha_1 \in (0, 2))$:

- Relativistic stable process $(f(r) \approx r^{-\alpha}(1 + r)^{(\alpha+d-1)/2} e^{-r})$, for $d \geqslant 1$.

- Truncated stable process $(f(r) = r^{-\alpha} \mathbf{1}_{(0,1)}(r))$.

- Tempered stable process $(f(r) = r^{-\alpha} e^{-r})$.

- Isotropic Lamperti stable process $(f(r) = re^{\delta r}(e^s - 1)^{-\alpha-1}, \delta < \alpha + 1)$.

- Layered stable process $(f(r) = r^{-\alpha} \mathbf{1}_{(0,1)}(r) + r^{-\alpha_1} \mathbf{1}_{[1,\infty)}(r))$.

The scale invariant Harnack inequality for all these examples are known, for instance by [6], but to our best knowledge the global one only for the last one (see e.g. [7]). Another example to which our result applies is $f(r) = r^{-2} \log^{-2}(1 + r^{-\delta})$, for $\delta < 1$. Note that f does not satisfy the condition (1.5) in [6], so the scale invariant Harnack inequality can not be concluded from [6].

Example 3. Let ϕ be a Bernstein function comparable with a function regularly varying at infinity with index α. If $\alpha \in (0, 1]$ the scale invariant Harnack inequality holds for the corresponding subordinate Brownian motion. This covers for instance results of [16], where a particular $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1$ was considered, for which we even have the global scale invariant Harnack inequality due to Theorem 1.

Example 4. Let ψ_1 satisfy $\mathrm{WLSC}(\beta_1, 0, c_1)$ and ψ_1 satisfy $\mathrm{WLSC}(\beta_2, 0, c_2)$. Then $\psi_1 + \psi_2$ satisfies $\mathrm{WLSC}(\beta_1 \wedge \beta_2, 0, c_1 \wedge c_2)$. Hence, if ψ_1, ψ_2 are Lévy-Khinchine exponents of isotropic unimodal Lévy processes, then the global scale invariant Harnack inequality holds with constant depending only on dimension, $\beta_1 \wedge \beta_2$ and $c_1 \wedge c_2$. In particular the global scale invariant Harnack inequality holds for a sum of two independent isotropic α-stable process with exponents $\psi_1(x) = b_1 |x|^{\alpha_1}$ and $\psi_2(x) = b_2 |x|^{\alpha_2}$, where $0 < \alpha_1 \leqslant \alpha_2 \leqslant 2$. Moreover, the constant in the Harnack inequality depends only on dimension and α_1 in this case.

Example 5. Let X_t be an isotropic unimodal Lévy process with the characteristic exponent ψ, independent of a Brownian motion B_t, then the scale invariant Harnack inequality holds for $X_t + aB_t$, $a > 0$. If additionally ψ satisfies the following

$$\psi(x) \leqslant C\kappa^{-\beta}\psi(\kappa x), \qquad |x| \leqslant 1, \, \kappa < 1,$$

for some constants C and $\beta > 0$, then the global scale invariant Harnack inequality holds for $X_t + aB_t$, $a > 0$.

Of course for all of the above examples Hölder continuity for bounded harmonic functions holds as well.

4   Applications to more general Lévy Processes

Let X_t be a general Lévy process and $d \geqslant 3$. In this section we relax the assumptions and comment on validity of the previous results in this new setting.

We set three conditions which to some extent replace the core assumption of the previous section that the process is isotropic unimodal.

(A1) Assume that $\nu(dx) = \nu(x)dx$ and there exist constants C_1^*, $R > 0$ such that

$$\nu(x - y) \leqslant C_1^* r^{-d} \int_{B(x,r)} \nu(y - z)dz, \qquad \text{for any } r < |x - y|/2 \wedge R.$$

(A2) Assume that $G(dx) = G(x)dx$, $x \neq 0$, and there are constants C_2^*, $R > 0$ such that $(C_2^*)^{-1}\tilde{G}(|x|) \leqslant G(x) \leqslant C_2^*\tilde{G}(|x|)$, for $|x| \leqslant R$ and $\tilde{G}$ is non-increasing.

(A3) There exists a constant C_3^* such that $|\operatorname{Im}\psi(x)| \leqslant C_3^* \operatorname{Re}\psi(x)$ and $\psi^*(|x|) \leqslant C_3^* \operatorname{Re}\psi(x)$, $x \in \mathbb{R}^d$.

Notice that under (A3) process is transient ($d \geqslant 3$).

In Remark 8 we explain that the claim of Corollary 7 holds if $|\operatorname{Im}\psi(x)| \leqslant C_3^* \operatorname{Re}\psi(x)$, $x \in \mathbb{R}^d$. Of course then the comparability constant will be depend on C_3^*. This condition is also sufficient to get (13). The second claim of Corollary 9 holds if we assume additionally that $\operatorname{Re}\psi$ satisfies $\mathrm{WLSC}(\beta, \theta, C)$. If we assume (A3) we infer the claim of Lemma 12. Indeed, under (A3) we have

$$\lambda \mathcal{L}f(\lambda) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4} \frac{\operatorname{Re}\psi(x)}{|\psi(x)|^2}dx \approx \int_{\mathbb{R}^d} e^{-|x|^2/4} \frac{dx}{\psi^*(|x|)}.$$

In the proof of Proposition 13 we used only Lemma 12 and Lemma 3, hence the conclusion of Proposition 13 holds under (A3).

In the proof of a counterpart of Proposition 14 and (16) we use the following theorem.

Theorem 29. *([11], Theorem 3.3) Let X_1 and X_2 be Lévy processes having exponents ψ_1 and ψ_2, and capacities Cap_1^λ and Cap_2^λ respectively. If $\lambda > 0$ and $C > 0$ are such that*

$$\mathrm{Re}\left(\frac{1}{\lambda + \psi_1(x)}\right) \leqslant C\,\mathrm{Re}\left(\frac{1}{\lambda + \psi_2(x)}\right) \quad \text{for all } x,$$

then

$$\mathrm{Cap}_2^\lambda(A) \leqslant 4C\,\mathrm{Cap}_1^\lambda(A)$$

for any analytic set A.

Lemma 30. *For any Lévy process X_t there exists a subordinate Brownian motion with the characteristic exponent $\phi(|x|^2)$ such that, for $r \geqslant 0$,*

$$\frac{1}{8(1+2d)}\phi(r^2) \leqslant \psi^*(r) \leqslant 4\phi(r^2).$$

Proof. Let us define a Bernstein function ϕ by the formula

$$\phi(r) = \int_{\mathbb{R}^d}\left(1 - e^{-|z|^2 r}\right)\nu(dz) + ||A||r.$$

Since $\frac{1}{2}\left(1 \wedge u\right) \leqslant 1 - e^{-u} \leqslant 1 \wedge u$

$$\frac{1}{2}\int_{\mathbb{R}^d}\left(1 \wedge (|z|r)^2\right)\nu(dz) + ||A||r^2 \leqslant \phi(r^2) \leqslant \int_{\mathbb{R}^d}\left(1 \wedge (|z|r)^2\right)\nu(dz) + ||A||r^2,$$

which completes the proof due to Lemma 6. $\qquad\square$

The following proposition is a counterpart of Proposition 14 and (16).

Proposition 31. *Let (A3) hold. Then*

$$\mathrm{Cap}(\overline{B}_r) \approx r^d \psi^*(r^{-1}).$$

Moreover, for any non-empty Borel set A we get

$$\mathrm{Cap}(A) \geqslant C(d, C_3^*)\psi^*(|A|^{-1/d})|A|.$$

Proof. Let $\lambda > 0$. Then, by (A3),

$$\frac{1}{(1 + (C_3^*)^2)(\lambda + \psi^*(|x|))} \leqslant \mathrm{Re}\left(\frac{1}{\lambda + \psi(x)}\right) \leqslant \frac{C_3^*}{\lambda + \psi^*(|x|)}.$$

Hence, by Theorem 29 and Lemma 30 we obtain, for any analytic set A,

$$\frac{1}{32(1+8d)C_3^*}\mathrm{Cap}_\phi^\lambda(A) \leqslant \mathrm{Cap}^\lambda(A) \leqslant 16(1 + (C_3^*)^2)\,\mathrm{Cap}_\phi^\lambda(A),$$

where $\mathrm{Cap}_\phi^\lambda$ denote the capacity of a subordinate Brownian motion Y_t with the characteristic exponent ϕ defined in Lemma 30. Since $\mathrm{Cap}(A) = \lim_{\lambda \to 0^+}\mathrm{Cap}^\lambda(A)$ the above inequality holds also for $\lambda = 0$. Finally, we use (14), (16) for Y_t and again Lemma 30 to get the conclusion. $\qquad\square$

To get conclusions of Theorem 16, Corollary 17 and Proposition 19 it is enough to assume (A2), (A3) and the weak lower scaling condition for $\operatorname{Re}\phi$. Under the same assumptions Proposition 25 holds. We additionally need to assume (A1) to prove Proposition 24. Finally we have the following theorems.

Theorem 32. *Suppose that (A1)-(A3) hold and $\operatorname{Re}\psi$ satisfies the weak lower scaling condition, then the scale invariant Harnack inequality holds.*

Theorem 33. *Suppose that (A1)-(A3) hold and $\operatorname{Re}\psi$ satisfies the weak lower scaling condition, then the conclusion of Theorem 2 hold.*

Remark 34. If (A1), (A2) holds for $R = \infty$ and $\operatorname{Re}\psi$ satisfies the global weak lower scaling condition then the global scale invariant Harnack inequality holds.

For instance, we can use our results to the sum of two independent isotropic stable processes with drift. More precisely we consider a process X_t with $\psi(x) = |x|^{\alpha_1} + |x|^{\alpha_2} - i\langle x, \gamma \rangle$, where $1 < \alpha_1 < 2$, $\alpha_2 \leqslant 1$ and $\gamma \in \mathbb{R}^d$. It is easy to see that this process satisfy (A1) and (A3) and the global weak lower scaling condition. Since estimates for the heat kernel of this process are known locally in time (see [24]) and estimates for the heat kernel of the sum of two independent isotropic stable process ([7]) one can check that potential kernels of these two processes are locally comparable. Hence the assumption (A2) is satisfied. Therefore we infer that the scale invariant Harnack inequality holds for X_t.

Note that similar condition as (A1) appeared in [4] and [1] and exactly the same in [5] and [6]. Instead of conditions (A2) and (A3) the authors of the above mentioned papers assumed some additional conditions for a Lévy measure.

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